

# Global well-posedness and scattering for the defocusing, mass - critical generalized KdV equation

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**Abstract:** In this paper we prove that the defocusing, mass - critical generalized KdV initial value problem is globally well-posed and scattering for  $u_0 \in L^2(\mathbf{R})$ . We prove this via a concentration compactness argument.

## 1 Introduction

In this paper we plan to study the global well - posedness theory for the initial value problem for the defocusing generalized KdV equation,

$$\partial_t u + \partial_{xxx} u = \partial_x(u^5), \quad u(0) \in L^2(\mathbf{R}), \quad x \in \mathbf{R}, t \in \mathbf{R}. \quad (1.1)$$

The set of solutions of (1.1) is invariant under the scaling

$$u_\lambda(x, t) = \lambda^{1/2} u(\lambda^3 t, \lambda x) \quad (1.2)$$

in the sense that if  $u$  solves (1.1) then so does  $u_\lambda$  with initial datum

$$u_\lambda(0, x) = \lambda^{1/2} u(0, \lambda x). \quad (1.3)$$

Notice that  $\|u_\lambda(0, x)\|_{L^2(\mathbf{R})} = \|u(0, x)\|_{L^2(\mathbf{R})}$ , so (1.1) is an  $L^2$  critical generalized KdV equation. The  $L^2$  norm, or mass, is conserved under the flow (1.1).

$$M(u(t)) = \int_{\mathbf{R}} |u(t, x)|^2 dx = M(u(0)). \quad (1.4)$$

Another conserved quantity of (1.1) is the energy

$$E(u(t)) = \frac{1}{2} \int_{-\infty}^{\infty} u_x^2(t, x) dx + \frac{1}{6} \int_{-\infty}^{\infty} u^6(t, x) dx = E(u(0)). \quad (1.5)$$

We define a solution of (1.1) to be a strong solution.

**Definition 1.1 (Solution)** A function  $u : I \times \mathbf{R} \rightarrow \mathbf{R}$  on a non - empty interval  $0 \in I \subset \mathbf{R}$  is a (strong) solution to (1.1) if it lies in the class  $C_t^0 L_x^2(J \times \mathbf{R}) \cap L_x^5 L_t^{10}(J \times \mathbf{R})$  for any compact  $J \subset I$ , and obeys the Duhamel formula

$$u(t) = e^{-t\partial_x^3} u_0 + \int_0^t e^{-(t-\tau)\partial_x^3} \partial_x(u^5(\tau)) d\tau. \quad (1.6)$$

We refer to the interval  $I$  as the lifespan of  $u$ . We say that  $u$  is a maximal lifespan solution if the solution cannot be extended to any strictly larger interval. We say that  $u$  is a global solution if  $I = \mathbf{R}$ .

[8] developed a global in time theory for initial data small enough in  $L_x^2(\mathbf{R})$ . The results turn local for arbitrary data with the time of existence depending on the shape of the initial data  $u_0$  not just its size. In particular, if  $u_0$  is a little bit more regular than  $L_x^2(\mathbf{R})$ , say  $u_0 \in H_x^s(\mathbf{R})$  for some  $s > 0$ , then a solution to (1.1) exists on a time interval  $[0, T]$ ,  $T(\|u_0\|_{H_x^s(\mathbf{R})}) > 0$ . This implies that a solution to (1.1) is global if  $u_0 \in H_x^1(\mathbf{R})$ .

From (1.1) we can see that it is important to analyze the scattering size.

**Definition 1.2 (Scattering size)**

$$S_I(u) = \int_{\mathbf{R}} \left( \int_I |u(t, x)|^{10} dt \right)^{1/2} dx = \|u\|_{L_x^5 L_t^{10}(I \times \mathbf{R})}^5. \quad (1.7)$$

Associated with the notion of a solution is a corresponding notion of blowup.

**Definition 1.3 (Blowup)** We say that a solution  $u$  to (1.1) blows up forward in time if there exists  $t_1 \in I$  such that

$$S_{[t_1, \sup(I))}(u) = \infty. \quad (1.8)$$

and that  $u$  blows up backward in time if there exists a time  $t_1 \in I$  such that

$$S_{(\inf(I), t_1]}(u) = \infty. \quad (1.9)$$

This precisely corresponds to the impossibility of continuing the solution (in the case of blowup in finite time) or failure to scatter (in the case of blowup in infinite time). We summarize the results of [8] below.

**Theorem 1.1 (Local well - posedness)** Given  $u_0 \in L_x^2(\mathbf{R})$  and  $t_0 \in \mathbf{R}$ , there exists a unique maximal lifespan solution  $u$  to (1.1) with  $u(t_0) = u_0$ . We will write  $I$  for the maximal lifespan. This solution also has the following properties:

1. (Local existence)  $I$  is an open neighborhood of  $t_0$ .

2. (Blowup criterion) If  $\sup(I)$  is finite then  $u$  blows up forward in time. If  $\inf(I)$  is finite then  $u$  blows up backward in time.

3. (Scattering) If  $\sup(I) = +\infty$  and  $u$  does not blow up forward in time, then  $u$  scatters forward in time. That is, there exists a unique  $u_+ \in L_x^2(\mathbf{R})$  such that

$$\lim_{t \rightarrow +\infty} \|u(t) - e^{-t\partial_x^3} u_+\|_{L_x^2(\mathbf{R})} = 0. \quad (1.10)$$

Conversely, given  $u_+ \in L_x^2(\mathbf{R})$  there is a unique solution to (1.1) in a neighborhood of  $\infty$  so that (1.10) holds. One can define scattering backward in time in a completely analogous manner.

4. (Small data global existence) If  $M(u_0)$  is sufficiently small then  $u$  is a global solution which does not blow up either forward or backward in time. Indeed, in this case

$$S_{\mathbf{R}}(u) \lesssim M(u)^{5/2}. \quad (1.11)$$

**Remark:** See [1] for the analogous result for the nonlinear Schrödinger equation. In this paper we will prove

**Theorem 1.2 (Spacetime bounds for the mass - critical gKdV)** *The defocusing mass - critical gKdV problem (1.1) is globally well - posed for arbitrary initial data  $u_0 \in L_x^2(\mathbf{R})$ . Furthermore, the global solution satisfies the following spacetime bounds*

$$\|u\|_{L_x^5 L_t^{10}(\mathbf{R} \times \mathbf{R})} \leq C(M(u_0)). \quad (1.12)$$

The function  $C : [0, \infty) \rightarrow [0, \infty)$ .

**Remark:** This paper does not consider the focusing problem at all. See [9] and [10] for more information on this topic and the conjectured result.

This theorem is proved using concentration compactness. [9] demonstrated that if a solution to (1.1) blows up in finite time  $T_* < \infty$ , there exists a  $C_0$  such that at least  $C_0$  amount of mass must concentrate in a window of width  $c(T_* - t)^{1/2} \|u(t)\|_{H_x^s}^{1/2s}$  for some  $s > 0$ .

Later, [10] proved a conditional concentration compactness result.

**Theorem 1.3 (Concentration compactness theorem)** *Assume that the defocusing mass - critical nonlinear Schrödinger equation in one dimension,*

$$(i\partial_t + \partial_{xx})v = |v|^4 v \quad (1.13)$$

has global spacetime bounds

$$\int_{\mathbf{R}} \int_{\mathbf{R}} |v(t, x)|^6 dx dt \leq C(M(v(0, x))). \quad (1.14)$$

Then if theorem 1.2 fails to be true, there exists a critical mass  $0 < M_c < \infty$  such that  $u$  is a blowup solution in both time directions to (1.1) on some maximal interval  $I$ ,  $M(u(t)) = M_c$ , and  $\{u(t) : t \in I\} \subset \{\lambda^{1/2} f(\lambda(x + x_0)) : \lambda \in (0, \infty), x_0 \in \mathbf{R}, f \in K\}$  for some compact  $K \subset L_x^2(\mathbf{R})$ .

Subsequently [3] proved that a solution to (1.13) does have the global spacetime bounds (1.14). Therefore, at this point it only remains to rule out the minimal mass blowup solution described in theorem 1.3. Notice that modulo symmetries in  $x_0$  and  $\lambda$  the minimal mass blowup solution described in theorem 1.3 lies in a precompact set. Therefore, a sequence of solutions will have a convergent subsequence modulo symmetries in  $x_0$  and  $\lambda$ . For any  $t \in I$  let  $N(t) \in (0, \infty)$  and  $x(t) \in \mathbf{R}$  be the scale function and spatial function respectively such that

$$N(t)^{-1/2} u(N(t)^{-1}(x - x(t))) \in K. \quad (1.15)$$

**Remark:** We have some flexibility with regard to the  $N(t)$ ,  $x(t)$  and  $K$  that we choose. This will be discussed in the concentration compactness section. To rule out the minimal mass blowup solution in theorem 1.3 it suffices to rule out one of three scenarios,

1. The self - similar scenario.

$$N(t) \sim t^{-1/3}, \quad t \in (0, \infty) \quad (1.16)$$

2. The double rapid cascade.

$$N(t) \geq 1, \quad N(0) = 1, \quad \int_I N(t)^2 dt \lesssim 1, \quad (1.17)$$

$$\lim_{t \nearrow \sup(I)} N(t) = \lim_{t \searrow \inf(I)} N(t) = +\infty. \quad (1.18)$$

3. The quasisoliton solution.

$$\int_J N(t)^3 dt \sim \mathcal{J}, \quad \int_J N(t)^2 \lesssim \mathcal{J}, \quad (1.19)$$

$$E(u(t)) \lesssim 1, \quad (1.20)$$

for some  $\mathcal{J}$  large,  $J \subset I$ .

The first two scenarios are precluded by an additional regularity argument. We use concentration compactness to show that in cases one and two  $E(u(t)) \lesssim 1$ , which prevents  $N(t) \nearrow \infty$ .

To rule out the quasisoliton we construct an interaction Morawetz estimate. We rely on the papers of [16] and then [13], which proved the nonexistence of a soliton solution to the generalized KdV equation by showing that the center of energy moves to the left faster than the center of mass. We utilize the computations in [16] to produce an interaction Morawetz estimate that is similar in flavor to the interaction Morawetz estimate of [4]. This rules out the final scenario, proving theorem 1.2.

In section two we discuss some properties of the linear solution to the Airy equation  $(\partial_t + \partial_{xxx})u = 0$  as well as estimates for the nonlinear equation (1.1). Most of these estimates can be found in [8] and [10]. We also will discuss the  $U_{\partial_x^3}$  and  $V_{\partial_x^3}$  spaces of [5].

In section three we will discuss the local conservation of the quantities mass and energy. We will use the computations of [16].

In section four we will describe the concentration compactness of [10]. We will then discuss our three minimal mass blowup scenarios.

In section five we will rule out the self - similar blowup scenario.

In section six we will rule out the double rapid cascade.

In section seven we will rule out the quasi - soliton.

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## 2 Linear Estimates

We are interested in the mixed norm spaces

$$L_x^p L_t^q(I \times \mathbf{R}) = \{F(x, t) : (\int_{\mathbf{R}} (\int_I |F(x, t)|^q dt)^{p/q} dx)^{1/p} < +\infty\}, \quad (2.1)$$

and

$$L_t^p L_x^q(I \times \mathbf{R}) = \{F(t, x) : (\int_I (\int_{\mathbf{R}} |F(t, x)|^q dx)^{p/q} dt)^{1/p} < +\infty\}. \quad (2.2)$$

**Definition 2.1**  $(p, q, \alpha)$  is an admissible triple if

$$\frac{1}{p} + \frac{1}{2q} = \frac{1}{4}, \quad \alpha = \frac{2}{q} - \frac{1}{p}, \quad 1 \leq p, q \leq \infty, \quad -\frac{1}{4} \leq \alpha \leq 1. \quad (2.3)$$

If  $(p, q, \alpha)$  is an admissible triple denote  $(p, q, \alpha) \in \mathcal{A}$ .

**Proposition 2.1 (Linear estimates)** Let  $u$  be a solution of the initial value problem

$$\begin{aligned} (\partial_t + \partial_x^3)u &= F, \quad t \in I, x \in \mathbf{R}, \\ u(0, x) &= u_0. \end{aligned} \quad (2.4)$$

Then for any admissible triples  $(p_j, q_j, \alpha_j)$ ,  $j = 1, 2$ ,

$$\|D_x^{\alpha_1} u\|_{L_x^{p_1} L_t^{q_1}(I \times \mathbf{R})} \lesssim \|u_0\|_{L^2(\mathbf{R})} + \|D_x^{-\alpha_2} F\|_{L_x^{p'_2} L_t^{q'_2}(I \times \mathbf{R})}. \quad (2.5)$$

*Proof:* This was proved in [9].  $\square$

Taking a cue from the analysis of the nonlinear Schrödinger equation (see for example [15]), consider the analogue of the Strichartz spaces in the gKdV case.

**Definition 2.2** Let

$$\|u\|_{S^0(I \times \mathbf{R})} = \sup_{(p, q, \alpha) \in \mathcal{A}} \|D_x^\alpha u\|_{L_x^p L_t^q(I \times \mathbf{R})}. \quad (2.6)$$

Then let  $N^0(I \times \mathbf{R})$  be the dual of  $S^0(I \times \mathbf{R})$  with appropriate norm.

$$\|F\|_{N^0(I \times \mathbf{R})} = \inf_{F=F_1+F_2} \|D_x^{1/4} F_1\|_{L_x^{4/3} L_t^1(I \times \mathbf{R})} + \|D_x^{-1} F_2\|_{L_x^1 L_t^2(I \times \mathbf{R})}. \quad (2.7)$$

**Lemma 2.2 (More linear estimates)** If  $u$  is a solution to (2.4) then

$$\|u\|_{S^0(I \times \mathbf{R})} + \|u\|_{L_t^\infty L_x^2(I \times \mathbf{R})} \lesssim \|u_0\|_{L_x^2(\mathbf{R})} + \|F_1\|_{N^0(I \times \mathbf{R})} + \|F_2\|_{L_t^1 L_x^2(I \times \mathbf{R})}, \quad (2.8)$$

for any  $F = F_1 + F_2$  decomposition.

*Proof:* See [6], [7], [8], and [9].  $\square$

In this paper it is useful to use the  $U_{\partial_x^3}^2$  and  $V_{\partial_x^3}^2$  spaces of [5].

**Definition 2.3** Let  $1 \leq p < \infty$ .  $u$  is a  $U_{\partial_x^3}^p(I \times \mathbf{R})$  atom if  $[t_0, t_1], [t_1, t_2], \dots$  is a partition of  $I$ ,

$$u = \sum_{t_j \nearrow} 1_{[t_j, t_{j+1}]}(t) e^{-t\partial_x^3} u(t_j), \quad (2.9)$$

$$\sum_{t_j \nearrow} \|u(t_j)\|_{L_x^2(\mathbf{R})}^p = 1. \quad (2.10)$$

Then define the norm

$$\|u\|_{U_{\partial_x^3}^p(I \times \mathbf{R})} = \inf \left\{ \sum_{\lambda} |c_{\lambda}| : \sum_{\lambda} c_{\lambda} u_{\lambda} = u \quad a.e., \quad u_{\lambda} \text{ is a } U_{\partial_x^3}^p \text{ atom} \right\}. \quad (2.11)$$

Let

$$\|v\|_{V_{\partial_x^3}^p(I \times \mathbf{R})}^p = \sup_{\{t_j \nearrow\}} \sum_{t_j \nearrow} \|e^{t_j \partial_x^3} v(t_j) - e^{t_{j+1} \partial_x^3} v(t_{j+1})\|_{L_x^2(\mathbf{R})}^p. \quad (2.12)$$

$$\|F\|_{DU_{\partial_x^3}^p(I \times \mathbf{R})} = \inf \{ \|u\|_{U_{\partial_x^3}^p(I \times \mathbf{R})} : (\partial_t + \partial_x^3)u = F \}. \quad (2.13)$$

**Remark:** By checking individual atoms and direct calculation,  $U_{\partial_x^3}^p \subset U_{\partial_x^3}^q$ ,  $V_{\partial_x^3}^p \subset V_{\partial_x^3}^q$  when  $p < q$ .

**Remark:** By checking individual atoms,

$$\|u\|_{S^0(I \times \mathbf{R})} \lesssim \|u\|_{U_{\partial_x^3}^2(I \times \mathbf{R})}. \quad (2.14)$$

It can be verified by direct calculation (see [5]) that

$$\|F\|_{DU_{\partial_x^3}^p(I \times \mathbf{R})} = \sup_{\|v\|_{V_{\partial_x^3}^{p'}(I \times \mathbf{R})} = 1} \langle v, F \rangle. \quad (2.15)$$

**Lemma 2.3** For a decomposition  $F = F_1 + F_2$ ,

$$\|F\|_{DU_{\partial_x^3}^2(I \times \mathbf{R})} \lesssim \| |\partial_x|^{-1/6} F_1 \|_{L_{t,x}^{6/5}(I \times \mathbf{R})} + \|F_2\|_{L_x^{5/4} L_t^{10/9}(I \times \mathbf{R})}. \quad (2.16)$$

$$\|\partial_x(u^5)\|_{DU_{\partial_x^3}^2(I \times \mathbf{R})} \lesssim \|u\|_{S^0(I \times \mathbf{R})}^5. \quad (2.17)$$

*Proof:* The first inequality follows from the embedding  $V_{\partial_x^3}^2 \subset U_{\partial_x^3}^p$  for any  $p > 2$  (see [5]). It can be verified by checking individual atoms that

$$\| |\partial_x|^{1/6} v \|_{L_{t,x}^6(I \times \mathbf{R})} + \|v\|_{L_x^5 L_t^{10}(I \times \mathbf{R})} \lesssim \|v\|_{U_{\partial_x^3}^5(I \times \mathbf{R})} \lesssim \|v\|_{V_{\partial_x^3}^2(I \times \mathbf{R})} = 1. \quad (2.18)$$

Next,

$$\|\partial_x(u^5)\|_{L_x^{5/4}L_t^{10/9}(I \times \mathbf{R})} \lesssim \|\partial_x u\|_{L_x^\infty L_t^2(I \times \mathbf{R})} \|u\|_{L_x^5 L_t^{10}(I \times \mathbf{R})}^4 \leq \|u\|_{S^0(I \times \mathbf{R})}^5. \quad (2.19)$$

This proves (2.17).  $\square$

We also make use of the dispersive estimate.

**Lemma 2.4 (Dispersive estimate)** *For  $2 \leq p \leq \infty$ ,*

$$\|e^{-t\partial_x^3} u_0\|_{L_x^p(\mathbf{R})} \lesssim t^{-\frac{2}{3}(\frac{1}{2}-\frac{1}{p})} \|u_0\|_{L_x^{p'}(\mathbf{R})}. \quad (2.20)$$

Finally it is useful to quote a long - time stability theorem.

**Theorem 2.5 (Long - time stability for the mass - critical gKdV)** *Let  $I$  be a time interval containing zero and let  $\tilde{u}$  be a solution to*

$$(\partial_t + \partial_{xxx})\tilde{u} = \partial_x(\tilde{u}^5) + e, \quad \tilde{u}(0, x) = \tilde{u}_0(x). \quad (2.21)$$

*Assume that*

$$\|\tilde{u}\|_{L_t^\infty L_x^2(I \times \mathbf{R})} \leq M, \|\tilde{u}\|_{L_x^5 L_t^{10}(I \times \mathbf{R})} \leq L \quad (2.22)$$

*for some positive constants  $M$  and  $L$ . Let  $u_0$  be such that*

$$\|u_0 - \tilde{u}_0\|_{L_x^2} \leq M'. \quad (2.23)$$

*Assume also the smallness conditions*

$$\begin{aligned} \|e^{-t\partial_x^3}(u_0 - \tilde{u}_0)\|_{L_x^5 L_t^{10}(I \times \mathbf{R})} &\leq \epsilon, \\ \|e\|_{N^0(I \times \mathbf{R})} &\leq \epsilon, \end{aligned} \quad (2.24)$$

*for some small  $0 < \epsilon < \epsilon_1(M, M', L)$ . Then there exists a solution  $u$  to (1.1) on  $I \times \mathbf{R}$  with initial data  $u_0$  at time  $t = 0$  satisfying*

$$\begin{aligned} \|u - \tilde{u}\|_{L_x^5 L_t^{10}(I \times \mathbf{R})} &\leq C(M, M', L)\epsilon, \\ \|u^5 - \tilde{u}^5\|_{L_x^1 L_t^2(I \times \mathbf{R})} &\leq C(M, M', L)\epsilon, \\ \|u - \tilde{u}\|_{L_t^\infty L_x^2(I \times \mathbf{R})} + \|u - \tilde{u}\|_{S^0(I \times \mathbf{R})} &\leq C(M, M', L), \\ \|u\|_{L_t^\infty L_x^2(I \times \mathbf{R})} + \|u\|_{S^0(I \times \mathbf{R})} &\leq C(M, M', L). \end{aligned} \quad (2.25)$$

*Proof:* See [10].  $\square$

In particular, this theorem implies that if  $u_0^n \rightarrow u_0$  strongly in  $L^2$ , and  $u$  is the solution to (1.1) on  $I \subset \mathbf{R}$  with initial data  $u_0$ , then for any  $J \subset I$ ,

$$\|u\|_{S^0(J \times \mathbf{R})} \leq C < \infty, \quad (2.26)$$

then  $u^n \rightarrow u$  in  $S^0(I \times \mathbf{R})$  and  $L_t^\infty L_x^2(I \times \mathbf{R})$ , where  $u^n$  is the solution to (1.1) with initial data  $u_0^n$ .

### 3 Local Conservation of mass and energy

In this section we list the local conservation laws used in many places, for example [16] and [13].

**Definition 3.1 (Mass density and mass current)** *The mass density is given by*

$$\rho(t, x) = u(t, x)^2. \quad (3.1)$$

*The mass current is given by*

$$j(t, x) = 3u_x(t, x)^2 + \frac{5}{3}u(t, x)^6. \quad (3.2)$$

**Definition 3.2 (Energy density and energy current)** *The energy density is given by*

$$e(t, x) = \frac{1}{2}u_x(t, x)^2 + \frac{1}{6}u(t, x)^6. \quad (3.3)$$

*The energy current is given by*

$$k(t, x) = \frac{3}{2}u_{xx}(t, x)^2 + 2u(t, x)^4u_x(t, x)^2 + \frac{1}{2}u(t, x)^{10}. \quad (3.4)$$

A routine computation verifies (for Schwartz solutions, at least) the pointwise conservation laws

$$\rho_t + \rho_{xxx} = j_x, \quad (3.5)$$

$$e_t + e_{xxx} = k_x. \quad (3.6)$$

In section seven we will make use of the monotonicity formula.

**Lemma 3.1 (Monotonicity formula)** *For a smooth function  $u$ ,*

$$\left(\int \rho(t, x) dx\right) \left(\int k(t, x) dx\right) - \left(\int e(t, x) dx\right) \left(\int j(t, x) dx\right) > 0. \quad (3.7)$$

*Proof:* See [16].  $\square$

**Remark:** Frequently in this paper it will be necessary to integrate by parts. This paper will always assume that the solution is smooth in space and time. An arbitrary solution can be well approximated by a smooth solution, and the bounds obtained will not depend on the smoothness of  $u$ . Similar computations are done in the case of the interaction Morawetz estimate for the Schrödinger equation. See for example [2].

## 4 Concentration Compactness

An important step in the study of the mass critical generalized KdV was the reduction of [10] to solutions that are almost periodic modulo symmetries.

**Definition 4.1 (Almost periodic modulo symmetries)** *A solution  $u$  to (the mKdV problem) with lifespan  $I$  is said to be almost periodic modulo symmetries if there exist functions  $N : I \rightarrow \mathbf{R}^+$ ,  $x : I \rightarrow \mathbf{R}$ ,  $C : \mathbf{R}^+ \rightarrow \mathbf{R}^+$  such that for all  $t \in I$  and  $\eta > 0$ ,*

$$\int_{|x-x(t)| \geq \frac{C(\eta)}{N(t)}} |u(t, x)|^2 dx + \int_{|\xi| \geq C(\eta)N(t)} |\hat{u}(t, \xi)|^2 d\xi < \eta. \quad (4.1)$$

$N$  will be called the frequency scale function for a solution  $u$ ,  $x$  the spatial center function, and  $C$  the compactness modulus function.

**Remark:** The parameter  $N(t)$  measures the frequency scale of the solution at time  $t$ , while  $\frac{1}{N(t)}$  measures the spatial scale. We can multiply  $N(t)$  by any function  $\alpha(t)$ ,  $0 < \epsilon < \alpha(t) < \frac{1}{\epsilon}$ , provided we also modify the compactness modulus function accordingly.

**Theorem 4.1 (Arzela - Ascoli theorem)** *A family of functions is precompact in  $L_x^2(\mathbf{R})$  if and only if it is norm bounded and there exists a compactness modulus function  $C$  such that*

$$\int_{|x| \geq C(\eta)} |f(x)|^2 dx + \int_{|\xi| \geq C(\eta)} |\hat{f}(\xi)|^2 d\xi \leq \eta \quad (4.2)$$

for all functions  $f$  in the family.

This implies that  $f$  is almost periodic modulo symmetries if and only if for some compact subset  $K \subset L_x^2(\mathbf{R})$ ,

$$\{u(t) : t \in I\} \subseteq \{\lambda^{1/2} f(\lambda(x + x_0)) : \lambda \in (0, \infty), x_0 \in \mathbf{R}, f \in K\}. \quad (4.3)$$

Let

$$L(M) = \sup\{S_I(u) : u : I \times \mathbf{R} \rightarrow \mathbf{R}, M(u) \leq M_c\}. \quad (4.4)$$

The supremum is taken over all solutions  $u : I \times \mathbf{R} \rightarrow \mathbf{R}$  obeying  $M(u) \leq M$ . For  $M$  small, a small data result implies  $L(M) \lesssim M^{5/2}$ . This fact combined with theorem 2.5 implies that failure of theorem 1.2 is equivalent to the existence of a critical mass  $M_c \in (0, \infty)$  such that

$$L(M) < \infty \text{ for } M < M_c, \quad L(M) = \infty \text{ for } M \geq M_c, \quad (4.5)$$

**Theorem 4.2** *Assume theorem 2.5 fails. Let  $M_c$  denote the critical mass. Then there exists a maximal lifespan solution to the mass - critical gKdV with mass  $M(u) = M_c$  which is almost periodic modulo symmetries and blows up both forward and backward in time. Also,  $[0, \infty) \subset I$ ,  $N(t) \leq 1$  for  $t \geq 0$ , and*

$$|N'(t)| \lesssim N(t)^4, \quad |x'(t)| \lesssim N(t)^2. \quad (4.6)$$

Moreover, there exists  $\delta(u) > 0$  such that for any  $t_0 \in I$ ,

$$\|u\|_{S^0([t_0, t_0 + \frac{\delta}{N(t_0)^3}] \times \mathbf{R})} \lesssim 1. \quad (4.7)$$

*Proof:* See [10]. The proof of theorem 4.2 was conditional on the assumption that the following mass - critical nonlinear Schrödinger equation result was true.  $\square$

**Lemma 4.3 (No waste lemma)** *If  $u$  is a minimal mass blowup solution to (1.1) then for any  $t \in I$ ,*

$$u(t) = \lim_{T \rightarrow \sup(I)} \int_t^T e^{-(t-\tau)\partial_x^3} \partial_x(u^5)(\tau) d\tau = \lim_{T \rightarrow \inf(I)} \int_t^T e^{-(t-\tau)\partial_x^3} \partial_x(u^5)(\tau) d\tau, \quad (4.8)$$

*weakly in  $L_x^2(\mathbf{R})$ .*

*Proof:* This follows in a similar manner to [17]. If  $\sup(I) = +\infty$  then  $N(t) \rightarrow +\infty$  combined with (4.1) implies

$$\lim_{T \rightarrow \sup(I)} \langle e^{-(t-T)\partial_x^3} u(T), u(t) \rangle = 0. \quad (4.9)$$

The same would be true if  $N(T) \rightarrow 0$ . If  $N(T) \sim N(t)$  as  $T \rightarrow \sup(I)$  then  $\sup(I) = +\infty$ . The dispersive estimate (2.20) combined with (4.1) implies that in this case also

$$\lim_{T \rightarrow \sup(I)} \langle e^{-(t-T)\partial_x^3} u(T), u(t) \rangle = 0. \quad (4.10)$$

$\square$

**Theorem 4.4** *If  $u$  is a solution to the one dimensional, mass - critical nonlinear Schrödinger equation*

$$(i\partial_t + \partial_{xx})u = |u|^4u, \quad (4.11)$$

*Then*

$$\|u\|_{L_{t,x}^6(\mathbf{R} \times \mathbf{R})} \leq C(\|u(0, \cdot)\|_{L^2}). \quad (4.12)$$

*Proof:* See [3].  $\square$

**Remark:** At this point we will select one minimal mass blowup solution in the form of theorem 4.2 and then show that this solution does not exist. Therefore we can abbreviate  $A \leq C(u)B$  as  $A \lesssim B$ .

We rule out three separate scenarios. Let

$$t_0(T) = \inf\{t \in [0, T] : N(t) = \inf_{t \in [0, T]} N(t)\} \quad (4.13)$$

$N(t)$  attains its infimum on  $[0, T]$  since  $N(t)$  is continuous.

**Case 1:** Self - similar solution.

$$\limsup_{T \rightarrow \infty} \left( \inf_{t \in [0, T]} N(t) \right) \cdot \left( \int_0^T N(t)^2 dt \right) \leq C < +\infty. \quad (4.14)$$

$$\limsup_{T \rightarrow \infty} \frac{\sup_{t \in [t_0(T), T]} N(t)}{N(t_0(T))} \leq C < +\infty. \quad (4.15)$$

**Case 2:** Rapid double cascade.

$$\limsup_{T \rightarrow \infty} \left( \inf_{t \in [0, T]} N(t) \right) \cdot \left( \int_0^T N(t)^2 dt \right) = C < \infty. \quad (4.16)$$

$$\limsup_{T \rightarrow \infty} \frac{\sup_{t \in [t_0(T), T]} N(t)}{N(t_0(T))} = +\infty. \quad (4.17)$$

**Case 3:** Quasi - soliton.

$$\limsup_{T \rightarrow \infty} \left( \inf_{t \in [0, T]} N(t) \right) \left( \int_0^T N(t)^2 dt \right) = +\infty. \quad (4.18)$$

## 5 Self - Similar solution

(4.14) implies

$$\liminf_{t \rightarrow \infty} N(t) = 0. \quad (5.1)$$

Then (4.15) implies that  $N(t) \rightarrow 0$  as  $t \rightarrow \infty$ . For any integer  $l \geq 0$  let

$$t_l = \inf\{t : N(t) = 2^{-l}\}. \quad (5.2)$$

Clearly  $t_0 = 1$ . By (4.14)

$$2^{-l} t_l 2^{-2l} \leq C, \quad (5.3)$$

so for any  $l$ ,  $t_l \lesssim 2^{3l}$ . On the other hand  $|N'(t)| \lesssim N(t)^4$  and (4.15) imply

$$2^{-l} \leq \int_{t_{l-1}}^{t_l} |N'(t)| dt \lesssim 2^{-4l} (t_l - t_{l-1}) \leq 2^{-4l} t_l. \quad (5.4)$$

This implies  $t_l \gtrsim 2^{3l}$  and therefore  $t_l \sim 2^{3l}$ , so for  $t \geq 1$ , (4.15) implies that  $N(t) \sim t^{-1/3}$ . Possibly after modifying  $C(\eta)$  by a constant, let  $N(t) = 1$  for  $t \in [0, 1]$ ,  $N(t) = t^{-1/3}$  for  $t \in [1, \infty)$ .

Let  $x(0) = 0$ .  $|x'(t)| \lesssim N(t)^2$  so  $|x(t)| \lesssim t^{1/3}$ . Therefore, again after modifying  $C(\eta)$  by a constant, for any  $\eta > 0$  there exists  $C(\eta) < \infty$  such that

$$\int_{|x| \geq \frac{C(\eta)}{N(t)}} u(t, x)^2 dx + \int_{|\xi| \geq C(\eta)N(t)} |\hat{u}(t, \xi)|^2 d\xi < \eta. \quad (5.5)$$

Now take a sequence  $t_n \rightarrow +\infty$  and let

$$u_0^n = \frac{1}{N(t_n)^{1/3}} u\left(\frac{x}{N(t_n)}\right). \quad (5.6)$$

Then, passing to a subsequence,  $u_0^n \rightarrow u_0$  in  $L^2$  and if  $u(1, \cdot) = u_0(\cdot)$ ,  $u$  solves the mass critical mKdV, then  $u$  is a self - similar blowup solution on  $(0, \infty)$  and  $N(t) = t^{-1/3}$ . We then prove

**Theorem 5.1 (Additional regularity)** *If  $u$  is a self - similar solution to the mass critical gKdV equation then  $u(1) \in H_x^1(\mathbf{R}) \cap L^6(\mathbf{R})$ .*

**Corollary 5.2 (No self - similar solution)** *There does not exist a self - similar solution.*

*Proof:* conservation of energy contradicts  $N(t) \rightarrow +\infty$  as  $t \rightarrow 0$ .

*Proof of theorem 5.1:* This proof is very similar to the additional regularity proof in [12], [11], and [18] for the self - similar blowup solution for the nonlinear Schrödinger equation. The proof has two steps. First, using the double Duhamel formula we prove that a self - similar solution must possess some additional regularity. More precisely, for some  $s > 0$ ,

$$\|u\|_{H_x^s(\mathbf{R})} \sim t^{-s/3}. \quad (5.7)$$

Then we argue by induction to show that in fact  $u \in H_x^1(\mathbf{R})$ . Let

$$\mathcal{M}(A) = \sup_T \|u_{\geq AT^{-1/3}}\|_{L_t^\infty L_x^2([T, 2T] \times \mathbf{R})}, \quad (5.8)$$

$$\mathcal{S}(A) = \sup_T \|u_{\geq AT^{-1/3}}\|_{U_{\partial_x^3}([T, 2T] \times \mathbf{R})}, \quad (5.9)$$

$$\mathcal{N}(A) = \sup_T \|P_{\geq AT^{-1/3}} \partial_x(u^5)\|_{U_{\partial_x^3}([T, 2T] \times \mathbf{R})}. \quad (5.10)$$

By Duhamel's principle,

$$\mathcal{S}(A) \lesssim \mathcal{M}(A) + \mathcal{N}(A). \quad (5.11)$$

Compactness in  $L^2$  norm combined with the above estimate implies

$$\lim_{A \rightarrow \infty} \mathcal{M}(A) + \mathcal{S}(A) + \mathcal{N}(A) = 0. \quad (5.12)$$

Let  $\alpha(k)$  be a frequency envelope that bounds  $\|P_{2^k} u(1)\|_{L^2}$ . Set  $\delta = \frac{1}{40}$ . Let

$$\alpha(k) = \sum_j 2^{-\delta|j-k|} \|P_{2^j} u(1)\|_{L^2}. \quad (5.13)$$

Choose  $\epsilon > 0$  very small,  $k_0(\epsilon)$  sufficiently large so that

$$\mathcal{M}(2^{k_0/2}) + \mathcal{S}(2^{k_0/2}) + \mathcal{N}(2^{k_0/2}) < \epsilon, \quad (5.14)$$

$$2^{-k_0} < \epsilon^{200}, \quad (5.15)$$

$$\sum_{k > k_0/2} \alpha(k)^2 \leq \epsilon^2. \quad (5.16)$$

**Theorem 5.3** For  $k \geq k_0$ ,

$$\|P_{2^k} u\|_{U_{\partial_x^3}([1, 2^{6(k-k_0)}] \times \mathbf{R})} \lesssim \alpha(k), \quad (5.17)$$

and for  $j > 6(k - k_0)$ ,

$$\|P_{2^k} u\|_{U_{\partial_x^3}([2^j, 2^{j+1}] \times \mathbf{R})} \lesssim 2^{\frac{1}{10}(j-6(k-k_0))} \alpha(k). \quad (5.18)$$

*Proof:* We prove this by Duhamel's principle.

$$u(t) = e^{-(t-1)\partial_x^3} u(1) + \int_1^t e^{-(t-\tau)\partial_x^3} \partial_x(u^5) d\tau. \quad (5.19)$$

$$\|e^{-(t-1)\partial_x^3} P_{2^k} u(1)\|_{U_{\partial_x^3}([1, 2^{6(k-k_0)}] \times \mathbf{R})} \lesssim \alpha(k). \quad (5.20)$$

$$\|P_{2^k} \partial_x(u^5)\|_{DU_{\partial_x^3}([1, 2^{6(k-k_0)}] \times \mathbf{R})} \lesssim 2^{5k/6} \sum_{k_1 \geq k} \|P_{k_1} u\|_{L_{t,x}^6([1, 2^{6(k-k_0)}] \times \mathbf{R})}^5 \quad (5.21)$$

$$+ 2^{5k/6} \|P_{k-5 \leq \cdot \leq k+5} u\|_{L_x^\infty L_t^2([1, 2^{6(k-k_0-5)}] \times \mathbf{R})} \|P_{\leq k} u\|_{L_x^{24/5} L_t^{12}([1, 2^{6(k-k_0-5)}] \times \mathbf{R})}^4 \quad (5.22)$$

$$+ 2^{5k/6} \|P_{k-5 \leq \cdot \leq k+5} u\|_{L_x^\infty L_t^2([2^{6(k-k_0-5)}, 2^{6(k-k_0)}] \times \mathbf{R})} \|P_{\leq k} u\|_{L_x^{24/5} L_t^{12}([2^{6(k-k_0-5)}, 2^{6(k-k_0)}] \times \mathbf{R})}^4. \quad (5.23)$$

By the local smoothing estimates and the concentration compactness result, for  $j \geq k_0$ ,

$$\|P_{2^j} u\|_{L_x^{24/5} L_t^{12}([1, 2^{6(k-k_0)}] \times \mathbf{R})} \lesssim 2^{j/24} \|P_{2^j} u\|_{U_{\partial_x^3}([1, 2^{6(j-k_0)}] \times \mathbf{R})} + 2^{j/24} \|P_{2^j} u\|_{U_{\partial_x^3}([2^{6(j-k_0)}, 2^{6(k-k_0)}] \times \mathbf{R})} \quad (5.24)$$

$$\lesssim \alpha(j) 2^{j/24} + 2^{j/24} (k - j) \epsilon. \quad (5.25)$$

For  $k_0/2 \leq j \leq k_0$ ,

$$\|P_{2^j} u\|_{L_x^{24/5} L_t^{12}([1, 2^{6(k-k_0)}] \times \mathbf{R})} \lesssim 2^{j/24} \epsilon (k - k_0)^{5/24}. \quad (5.26)$$

Finally for  $j \leq k_0/2$ ,

$$\|P_{2^j} u\|_{L_x^{24/5} L_t^{12}([1, 2^{6(k-k_0)}] \times \mathbf{R})} \lesssim 2^{j/24} (k - k_0)^{5/24}. \quad (5.27)$$

Putting this all together,

$$2^{5k/6} \|P_{k-5 \leq \cdot \leq k+5} u\|_{L_x^\infty L_t^2([1, 2^{6(k-k_0-5)}] \times \mathbf{R})} \|P_{\leq k} u\|_{L_x^{24/5} L_t^{12}([1, 2^{6(k-k_0-5)}] \times \mathbf{R})}^4 \quad (5.28)$$

$$\begin{aligned} &\lesssim 2^{-k/6} \alpha(k) \left( \sum_{j \leq k} 2^{j/24} (\alpha(j) + \epsilon(k-j))^4 + 2^{-k/6} \alpha(k) \left( \sum_{j \leq k_0} 2^{j/24} \epsilon(k-k_0)^{5/24} \right)^4 \right. \\ &\quad \left. + 2^{-k/6} \alpha(k) \left( \sum_{j \leq k_0/2} 2^{j/24} (k-k_0)^{5/24} \right)^4 \right) \lesssim \alpha(k) \epsilon^4. \end{aligned} \quad (5.29)$$

Similarly,

$$2^{5k/6} \|P_{k-5 \leq \cdot \leq k+5} u\|_{L_x^\infty L_t^2([2^{6(k-k_0-5)}, 2^{6(k-k_0)}] \times \mathbf{R})} \|P_{\leq k} u\|_{L_x^{24/5} L_t^{12}([2^{6(k-k_0-5)}, 2^{6(k-k_0)}] \times \mathbf{R})}^4 \lesssim \alpha(k) \epsilon^4. \quad (5.30)$$

Finally,

$$2^{5k/6} \sum_{k_1 \geq k} \|P_{k_1} u\|_{L_{t,x}^6([1, 2^{6(k-k_0)}] \times \mathbf{R})}^5 \lesssim 2^{5k/6} \sum_{k_1 \geq k} \alpha(k_1)^5 2^{-5k_1/6} \lesssim \alpha(k) \epsilon^4. \quad (5.31)$$

Now take  $j > 6(k - k_0)$ .

$$\|P_k u\|_{U_{\partial_x^3}([2^j, 2^{j+1}] \times \mathbf{R})} \leq \|P_k u(2^j)\|_{L^2} + C 2^k \|P_k(u^5)\|_{DU_{\partial_x^3}([2^j, 2^{j+1}] \times \mathbf{R})} \quad (5.32)$$

$$\leq \|P_k u\|_{U_{\partial_x^3}([2^{j-1}, 2^j] \times \mathbf{R})} + C 2^k \|P_k(u^5)\|_{DU_{\partial_x^3}([2^j, 2^{j+1}] \times \mathbf{R})}. \quad (5.33)$$

By the same analysis as before,

$$2^k \|P_{k-5 \leq \cdot \leq k+5} u\|_{L_x^\infty L_t^2} \|P_{\leq k} u\|_{L_x^{24/5} L_t^{12}}^4 + 2^{5k/6} \sum_{k_1 > k} \|P_{k_1} u\|_{L_{t,x}^6}^5 \quad (5.34)$$

$$\lesssim 2^{(j-6(k-k_0))/10} \alpha(k) \epsilon^4 + 2^{5k/6} \epsilon^4 \sum_{k \leq k_1} 2^{(j-6(k-k_0))/10} 2^{-5k_1/6} \alpha(k_1) \lesssim 2^{(j-6(k-k_0))/10} \alpha(2^k) \epsilon^4. \quad (5.35)$$

Now make a bootstrapping argument. Let  $A$  be the set of  $T \in [1, \infty]$  such that for a large, fixed constant  $C$ ,

$$\|P_{2^k} u\|_{U_{\partial_x^3}([1, 2^{6(k-k_0)}] \cap [1, T] \times \mathbf{R})} \leq \frac{C}{2} \alpha(k), \quad (5.36)$$

and for  $j > 6(k - k_0)$ ,

$$\|P_{2^k} u\|_{U_{\partial_x^3}([2^j, 2^{j+1}] \cap [1, T] \times \mathbf{R})} \leq \frac{C}{2} 2^{\frac{1}{10}(j-6(k-k_0))} \alpha(k). \quad (5.37)$$

The set  $A$  is nonempty since  $1 \in A$ , and is closed. It remains to show that  $A$  is open. Suppose  $A = [1, T_0]$ . Then there exists  $T_0 < T < 2T_0$  such that

$$\|P_{2^k} u\|_{U_{\partial_x^3}([1, 2^{6(k-k_0)}] \cap [1, T] \times \mathbf{R})} \leq C\alpha(k), \quad (5.38)$$

and for  $j > 6(k - k_0)$ ,

$$\|P_{2^k} u\|_{U_{\partial_x^3}([2^j, 2^{j+1}] \cap [1, T] \times \mathbf{R})} \leq C 2^{\frac{1}{10}(j-6(k-k_0))} \alpha(k). \quad (5.39)$$

For  $\epsilon > 0$  sufficiently small,

$$\|P_{2^k} u\|_{U_{\partial_x^3}([1, 2^{6(k-k_0)}] \cap [1, T] \times \mathbf{R})} \lesssim \alpha(k) + \alpha(k)\epsilon^4, \quad (5.40)$$

and for  $j > 6(k - k_0)$ ,

$$\|P_{2^k} u\|_{U_{\partial_x^3}([2^j, 2^{j+1}] \cap [1, T] \times \mathbf{R})} \lesssim C 2^{\frac{1}{10}(j-6(k-k_0))} \alpha(k)\epsilon^4 + \frac{C}{2} 2^{\frac{1}{10}(j-1-6(k-k_0))} \alpha(k). \quad (5.41)$$

Choosing  $\epsilon > 0$  sufficiently small,  $C$  sufficiently large implies that the bounds for  $C$  imply the bounds for  $\frac{C}{2}$ , which closes the bootstrap, proving that  $A = [1, \infty)$ .  $\square$

Theorem 5.3 implies that for  $k > 6k_0$ ,

$$\|P_{>2^k} u\|_{U_{\partial_x^3}([2^{-5k/2}, 1] \times \mathbf{R})}^2 \lesssim \sum_{j \geq k} \left( \sum_{j_1} 2^{-\delta|j_1-j|} \|P_{j_1} u(2^{-5k/2})\|_{L_x^2(\mathbf{R})} \right)^2 \quad (5.42)$$

$$\begin{aligned} &\lesssim \sum_{j_1 \geq k - \frac{k_0}{2}} \|P_{j_1} u(2^{-5k/2})\|_{L_x^2(\mathbf{R})}^2 \sum_{j \geq k} 2^{-\delta|j_1-j|} + \sum_{j_1 \leq k - \frac{k_0}{2}} \|P_{j_1} u(2^{-5k/2})\|_{L_x^2(\mathbf{R})}^2 \sum_{j \geq k} 2^{-\delta|j_1-j|} \\ &\lesssim \epsilon^2 + \mathcal{M}(2^{k_0/2})^2 \lesssim \epsilon^2. \end{aligned} \quad (5.43)$$

By conservation of mass and the conditions on  $k_0$ .

Another useful fact about self - similar solutions is that a self - similar solution rescales to another self - similar solution. The scaling

$$u(t, x) \mapsto \frac{1}{\lambda^{1/2}} u\left(\frac{t}{\lambda^3}, \frac{x}{\lambda}\right) = u_\lambda(t, x) \quad (5.44)$$

with  $\lambda = 2^k$  rescales the self - similar solution to a new self - similar solution with

$$u_\lambda(1) = \frac{1}{\lambda^{1/2}} u\left(\frac{1}{\lambda^3}, \frac{x}{\lambda}\right). \quad (5.45)$$

The no - waste Duhamel formula (4.8) gives the double Duhamel formula

$$\|P_{2^k} u(1)\|_{L^2}^2 = \int_0^1 \int_1^\infty \langle e^{(t-\tau)\partial_x^3} P_k \partial_x(u^5), P_k \partial_x(u^5) \rangle dt d\tau. \quad (5.46)$$

$$\int_{2^{-5k/2}}^1 \int_1^{2^{6(k-k_0)}} \langle e^{(t-\tau)\partial_x^3} P_k \partial_x(u^5), P_k \partial_x(u^5) \rangle d\tau \quad (5.47)$$

$$\lesssim 2^{5k/3} \|P_k(u^5)\|_{L_{t,x}^{6/5}([1, 2^{6(k-k_0)}] \times \mathbf{R})} \|P_k(u^5)\|_{L_{t,x}^{6/5}([2^{-5k/2}, 1] \times \mathbf{R})}. \quad (5.48)$$

$$2^{5k/6} \|P_k(u^5)\|_{L_{t,x}^{6/5}([1, 2^{6(k-k_0)}] \times \mathbf{R})} \lesssim 2^{5k/6} \left( \sum_{j \geq k} \|P_j u\|_{L_{t,x}^6([1, 2^{6(k-k_0)}] \times \mathbf{R})}^5 \right) \quad (5.49)$$

$$+ 2^{-k/6} \|P_{k-5 \leq \cdot \leq k+5} u\|_{L_x^\infty L_t^2([1, 2^{6(k-k_0)}] \times \mathbf{R})} \left( \sum_{j \leq k} \|P_j u\|_{L_x^{24/5} L_t^{12}([1, 2^{6(k-k_0)}] \times \mathbf{R})} \right)^4. \quad (5.50)$$

For all  $j \geq k_0$ , theorem 5.3 implies

$$\|P_j u\|_{L_x^{24/5} L_t^{12}([1, 2^{6(k-k_0)}] \times \mathbf{R})} \lesssim \alpha(j) 2^{\frac{(k-j)}{10}} 2^{j/24}, \quad (5.51)$$

$$\|P_j u\|_{L_x^{24/5} L_t^{12}([1, 2^{6(k-k_0)}] \times \mathbf{R})} \lesssim \alpha(j) 2^{j/24} + 2^{j/24} (k-j)^{5/24}, \quad (5.52)$$

and for all  $j$ ,

$$\|P_j u\|_{L_x^{24/5} L_t^{12}([1, 2^{6(k-k_0)}] \times \mathbf{R})} \lesssim 2^{j/24} (k-k_0)^{5/24}. \quad (5.53)$$

Therefore,

$$2^{5k/6} \|P_k(u^5)\|_{L_{t,x}^{6/5}([1, 2^{6(k-k_0)}] \times \mathbf{R})} \lesssim \alpha(k)^2 + 2^{-k/6} \alpha(k) \sum_{k_0 \leq j \leq k} \alpha(j) 2^{(k-j)/10} 2^{j/6} (k-j+1)^{5/8} \quad (5.54)$$

$$+ 2^{-k/6} \alpha(k) \sum_{j \leq k_0} (k-k_0)^{5/6} 2^{j/6} \lesssim \alpha(k)^2 + 2^{-k/4}. \quad (5.55)$$

Also by (5.43) and the proof of theorem 5.3,

$$2^{5k/6} \|P_{>2^k}(u^5)\|_{L_{t,x}^{6/5}([2^{-5k/2}, 1] \times \mathbf{R})} \lesssim \epsilon^5. \quad (5.56)$$

Therefore,

$$(5.47) \lesssim (\alpha(k)^2 + 2^{-k/4})\epsilon^5. \quad (5.57)$$

Next,

$$\int_0^{2^{-5k/2}} \int_1^{2^{6(k-k_0)}} \langle e^{(t-\tau)\partial_x^3} P_k \partial_x(u^5), P_k \partial_x(u^5) \rangle dt d\tau \quad (5.58)$$

$$\lesssim 2^{2k} \|P_k(u^5)\|_{L_{t,x}^1([0, 2^{-5k/2}] \times \mathbf{R})} \left( \int_{2^{6(k-k_0)}}^\infty \frac{1}{t^{1/3}} \|P_k(u^5)\|_{L_x^1} dt \right). \quad (5.59)$$

$$\begin{aligned} & 2^k \|P_k(u^5)\|_{L_{t,x}^1([T, 2T] \times \mathbf{R})} \\ & \lesssim 2^k \|P_{>k-5} u\|_{L_x^\infty L_t^2} \|u\|_{L_t^\infty L_x^2}^{1/2} \|P_{\leq 2^k} u\|_{L_x^{14/3} L_t^{14}}^{7/2} + 2^k \|P_{>2^k} u\|_{L_{t,x}^6}^{9/2} \|u\|_{L_t^\infty L_x^2}^{1/2} \lesssim T^{1/4} 2^{k/4}. \end{aligned} \quad (5.60)$$

Therefore,

$$2^k \int_{2^{6(k-k_0)}}^\infty \frac{1}{t^{1/3}} \|P_{2^k}(u^5)\|_{L_x^1} dt \lesssim 2^{-k/4}. \quad (5.61)$$

By Holder's inequality,

$$2^k \|P_k(u^5)\|_{L_{t,x}^1([0, 2^{-5k/2}] \times \mathbf{R})} \lesssim 2^k \|u\|_{L_t^\infty L_x^2} \sum_{T < 2^{-5k/2}} T^{1/2} \|u\|_{L_{t,x}^8([T, 2T] \times \mathbf{R})}^4 \lesssim 2^{-k/4}. \quad (5.62)$$

Therefore,

$$\|P_k u(1)\|_{L^2}^2 \lesssim \epsilon^5 \alpha(k)^2 + 2^{-k/2} + 2^{-k/8} \epsilon^5 \alpha(k). \quad (5.63)$$

Let  $\beta(k)$  be another frequency envelope.

$$\beta(k) = \sum_j 2^{-\frac{\delta}{2}|j-k|} \|P_{2^k} u(1)\|_{L^2(\mathbf{R})}. \quad (5.64)$$

$$\begin{aligned} & \sum_j 2^{-\delta|j-k|/2} \|P_j u(1)\|_{L^2} \\ & \lesssim \epsilon^{5/2} \sum_j 2^{-\delta|j-k|/2} \sum_{j_1} 2^{-\delta|j-j_1|} \|P_{j_1} u(1)\|_{L^2} + \sum_j 2^{-\delta|j-k|/2} 2^{-j/8} \end{aligned} \quad (5.65)$$

implies that for  $\epsilon > 0$  sufficiently small,

$$\beta(k) \lesssim \epsilon^{5/2} \beta(k) + 2^{-\delta k/16}. \quad (5.66)$$

This implies after making the rescaling argument that

$$\mathcal{M}(2^k) \lesssim 2^{-\delta k/16}. \quad (5.67)$$

Now suppose that for some  $\sigma > 0$ ,

$$\mathcal{M}(A) \lesssim A^{-\sigma}. \quad (5.68)$$

$$\|P_{>AT^{-1/3}}\partial_x(u^5)\|_{DU_{\partial_x^3}([T,2T]\times\mathbf{R})} \lesssim \|P_{>AT^{-1/3}}u\|_{U_{\partial_x^3}([T,2T]\times\mathbf{R})}^5 \quad (5.69)$$

$$+ A^{-1/6}T^{1/18}\|\partial_x P_{>\frac{A}{32}T^{-1/3}}u\|_{L_x^\infty L_t^2([T,2T]\times\mathbf{R})}\|P_{2^{k_0/2}T^{-1/3}<.\frac{A}{32}T^{-1/3}}u\|_{L_x^{24/5}L_t^{12}([T,2T]\times\mathbf{R})}^4 \quad (5.70)$$

$$+ A^{-1/6}T^{1/18}\|\partial_x P_{>\frac{A}{32}T^{-1/3}}u\|_{L_x^\infty L_t^2([T,2T]\times\mathbf{R})}\|P_{\leq 2^{k_0/2}T^{-1/3}}u\|_{L_x^{24/5}L_t^{12}([T,2T]\times\mathbf{R})}^4 \lesssim \mathcal{S}(\frac{A}{32})\epsilon^4 + 2^{\frac{k_0}{2}}A^{-1/6}\mathcal{S}(\frac{A}{32}). \quad (5.71)$$

Since  $\mathcal{S}(A) \lesssim \mathcal{M}(A) + \mathcal{N}(A)$ ,

$$\mathcal{S}(A) \lesssim A^{-\sigma} + \mathcal{S}(\frac{A}{32})\epsilon^4 + 2^{k_0/12}A^{-1/6}\mathcal{S}(\frac{A}{32}), \quad (5.72)$$

so for  $A \geq 2^{k_0}$ , starting from  $\mathcal{S}(2^{k_0}) \leq \epsilon$ , by induction, taking  $\epsilon(\sigma) > 0$  sufficiently small, here it suffices to consider  $0 \leq \sigma \leq 2$ ,

$$\mathcal{S}(A) \lesssim A^{-\sigma} \quad (5.73)$$

which in turn implies

$$\mathcal{N}(A) \lesssim A^{-\sigma}. \quad (5.74)$$

Now we again use the no - waste Duhamel formula (2.20).

$$\|P_N u(1)\|_{L^2} \lesssim \sum_{k \geq 0} \|P_N(u^5)\|_{L_x^1 L_t^2([2^k, 2^{k+1}]\times\mathbf{R})} \lesssim \sum_{k \geq 0} \|P_{>\frac{N}{32}}u\|_{L_x^5 L_t^{10}([2^k, 2^{k+1}]\times\mathbf{R})}^5 \quad (5.75)$$

$$+ \sum_{k \geq 0} \|P_{>\frac{N}{32}}u\|_{L_x^\infty L_t^2([2^k, 2^{k+1}]\times\mathbf{R})}\|P_{\leq N}u\|_{L_x^4 L_t^\infty([2^k, 2^{k+1}]\times\mathbf{R})}^4 \quad (5.76)$$

$$\lesssim (N^{-5\sigma} + N^{-1-\sigma}) \sum_{k \geq 0} 2^{-k\sigma} \lesssim N^{-5\sigma} + N^{-1-\sigma}. \quad (5.77)$$

Iterating this argument finitely many times, this proves that  $u(1) \in H^1$ . This completes the proof of theorem 5.1.  $\square$

## 6 Rapid double cascade

**Theorem 6.1** *There does not exist a minimal mass blowup solution to the mass - critical gKdV in the form of a rapid double cascade.*

*Proof:* Let  $t_0 = t_0(T)$ , where  $t_0(T)$  is given by (4.13). Let

$$u_0^n = \frac{1}{N(t_0)^{1/2}} u(t_0, \frac{x + x(t_0)}{N(t_0)}). \quad (6.1)$$

By concentration compactness  $u_0^n$  has a subsequence that converges in  $L^2$  to  $u_0 \in L^2$ , and  $u_0$  is the initial data for a minimal mass blowup solution to the mKdV on a maximal interval  $I$ ,  $N(0) = 1$ ,  $N(t) \geq 1$  on  $I$ , and

$$\int_I N(t)^2 dt \lesssim C. \quad (6.2)$$

Since  $N(t) \geq 1$  this implies  $|I| \lesssim C$ , and also

$$\lim_{t \nearrow \sup(I)} N(t) = \lim_{t \searrow \inf(I)} N(t) = +\infty. \quad (6.3)$$

Since  $|x'(t)| \lesssim N(t)^2$  and  $x(0) = 0$ ,  $|x(t)| \lesssim C$  on  $I$ . Now define a Morawetz potential. Let  $\psi \in C^\infty(\mathbf{R})$ ,  $\psi$  is an odd function,  $\psi(x) = x$  for  $0 \leq x \leq 1$ ,  $\psi(x) = \frac{3}{2}$  for  $x > 1$ . Also let

$$0 \leq \phi(x) = \psi'(x). \quad (6.4)$$

For some  $0 < R < \infty$  let

$$M(t) = R \int \psi(\frac{x}{R}) u(t, x)^2 dx. \quad (6.5)$$

For any  $R > 0$ ,  $N(t) \nearrow \infty$ , as  $t \rightarrow \sup(I), \inf(I)$ , there exists  $t_+$  sufficiently close to  $\sup(I)$ ,  $t_-$  sufficiently close to  $\inf(I)$ , such that

$$R \int \psi(\frac{x}{R}) u(t_\pm, x)^2 dx \lesssim C. \quad (6.6)$$

Taking a derivative in time,

$$\frac{d}{dt} M(t) = - \int \phi(\frac{x}{R}) [3u_x^2 + \frac{5}{3}u^6] dx + O(\frac{1}{R^2}) \|u(t)\|_{L_x^2(\mathbf{R})}^2. \quad (6.7)$$

$$\frac{1}{R^2} \int_I \|u(t)\|_{L_x^2(\mathbf{R})}^2 dt \lesssim \frac{C}{R^2}. \quad (6.8)$$

Therefore, for any  $R > 1$

$$\int_I \int_{|x| \leq R} [3u_x^2 + \frac{5}{3}u^6] dx dt \lesssim C. \quad (6.9)$$

This bound is uniform in  $R$ , so in particular

$$\int_I \int [3u_x^2 + \frac{5}{3}u^6] dx dt \lesssim C. \quad (6.10)$$

(4.7) implies  $|I| \gtrsim 1$ . This in turn implies that there exists a  $t \in I$  such that

$$\int [3u_x^2 + \frac{5}{3}u^6] dx \lesssim C. \quad (6.11)$$

Conservation of energy then implies  $E(u(t)) = E(u(0)) \lesssim C$  for all  $t \in I$ , which contradicts  $N(t) \rightarrow +\infty$  as  $t \rightarrow \sup(I)$  or  $\inf(I)$ .  $\square$

## 7 Quasi - soliton

Let

$$R(T) = C \left( \int_0^T N(t)^2 dt \right) \quad (7.1)$$

for some fixed constant  $C$  such that  $|x'(t)| \leq \frac{C}{2} N(t)^2$ . (4.18) implies

$$\sup_{t \in [0, T]} \int_{|x| \geq R(T)} u(t, x)^2 dx \rightarrow 0, \quad (7.2)$$

as  $T \rightarrow \infty$ . Once again let

$$M(t) = R \int \psi\left(\frac{x}{R}\right) u(t, x)^2 dx. \quad (7.3)$$

$$M(T) - M(0) \lesssim R. \quad (7.4)$$

$$\dot{M}(t) = - \int \phi\left(\frac{x}{R}\right) [3u_x^2 + \frac{5}{3}u^6] dx + \frac{1}{R^2} \int \phi''\left(\frac{x}{R}\right) u^2 dx. \quad (7.5)$$

For any  $t_0 \in [0, T]$ ,

$$\int_{t_0}^{t_0 + \frac{\delta}{N(t_0)^3}} 1 dt = \frac{\delta}{N(t_0)^3} \lesssim \left( \int_{t_0}^{t_0 + \frac{\delta}{N(t_0)^3}} N(t)^2 dt \right). \quad (7.6)$$

This implies that since  $N(0) = 1$ ,

$$\int_0^T 1 dt \lesssim (\int_0^T N(t)^2 dt)^3 \quad (7.7)$$

which implies

$$\frac{1}{R^2} \int_0^T \int u(t, x)^2 dx dt \lesssim \int_0^T N(t)^2 dt. \quad (7.8)$$

Fix  $\mathcal{J} > 0$  large.

**Lemma 7.1** *There exists  $I(T) \subset [0, T]$  with*

$$\int_I N(t)^3 dt = \mathcal{J}, \quad \int_I \int_{|x| \leq R(T)} [3u_x^2 + \frac{5}{3}u^6] dx dt \lesssim \int_I N(t)^2 dt. \quad (7.9)$$

*The constant is uniform in  $T$ .*

Take  $[0, T]$  such that

$$\int_0^T N(t)^3 dt = K\mathcal{J} \quad (7.10)$$

for some integer  $K$ . Partition  $[0, T]$  into intervals  $I_j$ .

$$\sum_j \int_{I_j} \int_{|x| \leq R(T)} [3u_x^2 + \frac{5}{3}u^6] dx dt \lesssim \sum_j \int_{I_j} N(t)^2 dt. \quad (7.11)$$

Therefore there exists one  $j$  such that

$$\int_{I_j} \int_{|x| \leq R(T)} [3u_x^2 + \frac{5}{3}u^6] dx dt \lesssim \int_{I_j} N(t)^2 dt. \quad (7.12)$$

**Lemma 7.2** *There exists  $t_0(T) \in I(T)$  with*

$$N(t_0) \lesssim (\frac{1}{\mathcal{J}} \int_I N(t)^2 dt)^{-1}, \quad (7.13)$$

$$\int_{|x| \leq R(T)} [3u_x^2 + \frac{5}{3}u^6] dx \lesssim N(t_0)^2. \quad (7.14)$$

*Proof:* Suppose that for every  $t$  with  $N(t) \leq 10(\frac{1}{\mathcal{J}} \int N(t)^2 dt)^{-1}$ ,

$$\inf_{t \in J} \int_{|x| \leq R} [3u_x^2 + \frac{5}{3}u^6] dx \gg N(t)^2. \quad (7.15)$$

The contribution of these  $N(t)$ 's to  $\int N(t)^2 dt$  is small.

$$\int_{N(t) \geq 10(\frac{1}{\mathcal{J}} \int N(t)^2 dt)^{-1}} N(t)^2 \leq \frac{1}{10(\frac{1}{\mathcal{J}} \int_I N(t)^2 dt)^{-1}} \int_I N(t)^3 dt \leq \frac{1}{10} (\int_I N(t)^2 dt). \quad (7.16)$$

Therefore (7.15) implies

$$\int_I \int_{|x| \leq R(T)} [3u_x^2 + \frac{5}{3}u^6] dx dt >> \int_{I(T)} N(t)^2 dt, \quad (7.17)$$

which contradicts (7.12).  $\square$

The sequence

$$\chi(\frac{x}{R(T)}) \frac{1}{N(t_0(T))^{1/2}} u(\frac{x - x(t_0(T))}{N(t_0(T))}) \quad (7.18)$$

has a subsequence that converges in  $L^2$  to  $u_0 \in H^1$ ,  $E(u_0) \lesssim 1$ , and  $u_0$  is the initial data for a minimal mass blowup solution to the mKdV problem.

Moreover there exists an interval  $I$ ,  $0 \in I$ ,  $\int_I N(t)^3 dt = \mathcal{J}$  with

$$\int_I N(t)^2 dt \lesssim \int_I N(t)^3 dt \sim \mathcal{J}. \quad (7.19)$$

By Holder's inequality,

$$\mathcal{J}^2 \sim (\int_I N(t)^3 dt)^2 \lesssim (\int_I N(t)^2 dt) (\int_I N(t)^4 dt). \quad (7.20)$$

This implies that

$$\int_I N(t)^4 dt \gtrsim \mathcal{J}. \quad (7.21)$$

**Theorem 7.3 (No quasi - soliton)** *There does not exist a minimal mass blowup solution to (1.1) satisfying (7.19), (7.21),  $E(u(0)) \lesssim 1$  for  $\mathcal{J}$  sufficiently large.*

This theorem precludes the final minimal mass blowup solution since  $\int N(t)^3 dt$  is a scale invariant quantity and (4.7) implies that  $\int_I N(t)^3 dt = +\infty$ .

*Proof of theorem 7.3:* We follow [13], [3], and especially [16] to define an interaction Morawetz estimate. Recall (3.1) - (3.6). Define large constants  $R, R_1, R_1 \ll R$ . Let  $\chi_a \in C_0^\infty(\mathbf{R})$  be an even function,  $\chi_a = 1$  for  $|x| \leq a$ ,  $\chi_a = 0$  for  $|x| \geq a + R_1$ ,  $a \geq R$ . Let

$$\phi(x, y) = \frac{1}{R^2} \int_R^{2R} \int \chi_a(x-t) \chi_a(y-t) dt da. \quad (7.22)$$

$$\phi(x, y) = \frac{1}{R^2} \int_R^{2R} \int \chi_a(x-y-t) \chi_a(t) dt da = \phi(x-y) = \frac{1}{R^2} \int_R^{2R} \int \chi_a(y+t-x) \chi_a(-t) dt = \phi(y-x). \quad (7.23)$$

Then let

$$\psi(x-y) = \int_0^{x-y} \phi(t) dt. \quad (7.24)$$

Now we produce an interaction Morawetz estimate. Let

$$M(t) = R \int \int \psi\left(\frac{(x-y)\tilde{N}(t)}{R}\right) \rho(t, y)^2 e(t, x) dx dy. \quad (7.25)$$

$\tilde{N}(t)$  is a quantity,  $\tilde{N}(t) \leq N(t)$ , that will be defined shortly.

$$\dot{M}(t) = \tilde{N}(t) \int \int \phi\left(\frac{(x-y)\tilde{N}(t)}{R}\right) [-\rho(t, y)k(t, x) + j(t, y)e(t, x)] dx dy \quad (7.26)$$

$$+ \frac{\tilde{N}(t)^3}{R^2} \int \int \rho(t, y)^2 e(t, x)^2 dx dy \quad (7.27)$$

$$+ \int \int \frac{\tilde{N}'(t)(x-y)}{R} \phi\left(\frac{(x-y)\tilde{N}(t)}{R}\right) u(t, y)^2 \left[\frac{1}{2}u_x^2 + u^6\right] dx dy. \quad (7.28)$$

$$(7.26) = -\tilde{N}(t) \int \int \int \chi(x\tilde{N}(t)-s) \chi(y\tilde{N}(t)-s) [u(t, y)^2 \left(\frac{3}{2}u_{xx}^2 + 2u_x^2 u^4 + \frac{1}{2}u^{10}\right)] dx dy \quad (7.29)$$

$$+ \tilde{N}(t) \int \int \int \chi(x\tilde{N}(t)-s) \chi(y\tilde{N}(t)-s) \left[3u_y^2 + \frac{5}{3}u^6\right] \left[\frac{1}{2}u_x^2 + \frac{1}{6}u^6\right] dx dy \quad (7.30)$$

Let  $\tilde{\chi} = 1$  on  $[a, a+R_1]$  and 0 elsewhere. We will suppress the  $a$  for the moment and take  $\chi_a = \chi$  for some  $a$ .

$$\int \chi^2 u_{xx}^2 dx = \int \chi u_{xx} [\partial_{xx}(\chi u) - 2\chi_x u_x - \chi_{xx} u] dx \quad (7.31)$$

$$= \int \chi u_{xx} \partial_{xx}(\chi u) dx - \int \chi_x \chi \partial_x(u_x^2) dx - \int \chi_{xx} \chi u_{xx} u dx \quad (7.32)$$

$$= \int \partial_{xx}(\chi u)^2 dx - 2 \int \chi_x u_x \partial_{xx}(\chi u) dx - \int \chi_{xx} u \cdot \partial_{xx}(\chi u) dx \quad (7.33)$$

$$+ \int \frac{1}{2} \partial_{xx}(\chi^2) u^2 dx + \int \chi_{xx} \chi u_x^2 dx - \frac{1}{2} \int \partial_{xx}(\chi_{xx} \chi) u^2 dx$$

$$= \int \partial_{xx}(\chi u)^2 dx + \frac{1}{R_1^2} \int O(u_x^2 \tilde{\chi}^2) dx + \frac{1}{R_1^4} \int O(\tilde{\chi}^2 u^2) dx. \quad (7.34)$$

Next,

$$\int \chi^2 u_x^2 = \int \chi u_x \partial_x(\chi u) - \int \chi u_x \chi_x u \quad (7.35)$$

$$= \int \partial_x(\chi u)^2 + \frac{1}{4} \int \partial_{xx}(\chi^2) u^2 - \int \chi_x u \partial_x(\chi u) = \int \partial_x(\chi u)^2 + \frac{1}{R_1^2} \int \tilde{\chi}^2 u^2. \quad (7.36)$$

Next,

$$\int \chi^2 u_x^2 u^4 = \int \partial_x(\chi u) \chi u_x u^4 - \int \chi_x u \chi u_x u^4 \quad (7.37)$$

$$= \int \partial_x(\chi u)^2 u^4 - \int \chi_x u^5 \partial_x(\chi u) + \frac{1}{2} \int \partial_{xx}(\chi^2) u^6 \quad (7.38)$$

$$\geq \int \partial_x(\chi u)^2 (\chi u)^4 + \frac{1}{R_1^2} \int \tilde{\chi}^2 u^6. \quad (7.39)$$

Finally,

$$\int \chi^2 u^6 dx = \int (\chi u)^6 dx + \int (1 - \chi^4) (\chi u)^2 u^4. \quad (7.40)$$

From [16], if  $v = \chi_a u$ ,

$$\frac{3}{2} \left( \int v^2 \right) \left( \int v_{xx}^2 \right) - \frac{3}{2} \left( \int v_x^2 \right)^2 + 2 \left( \int v_x^2 v^4 \right) \left( \int v^2 \right) + \frac{1}{2} \left( \int v^{10} \right) \left( \int v^2 \right) \quad (7.41)$$

$$- \frac{4}{3} \left( \int v^6 \right) \left( \int v_x^2 \right) - \frac{1}{2} \left( \int v^6 \right)^2 > 0. \quad (7.42)$$

Next, for  $R$  sufficiently large, by Holder's inequality,

$$\frac{2}{9R} \int \left( \int \chi_a^6 \left( \frac{x \tilde{N}(t)}{R} - s \right) u(t, x)^6 dx \right)^2 ds \gtrsim \left( \int_{|x-x(t)| \leq \frac{C_0}{N(t)}} u(t, x)^6 dx \right) \gtrsim N(t)^4, \quad (7.43)$$

uniformly in  $a$ . Now we estimate the contribution of the errors. Let  $1_A(x)$  be the indicator function of a set  $A$ .

$$\frac{1}{R} \int_R^{2R} 1_{[a, a+R_1]} da \leq \frac{R_1}{R} 1_{[R, 3R]}. \quad (7.44)$$

By Holders inequality, Sobolev embedding, and (4.7),

$$\begin{aligned} \|u\|_{L_{t,x}^\delta([t_0, t_0 + \frac{\delta}{N(t_0)^3}] \times \mathbf{R})}^6 &\lesssim \|u_{\geq N(t_0)}\|_{L_{t,x}^\delta([t_0, t_0 + \frac{\delta}{N(t_0)^3}] \times \mathbf{R})}^6 + \|u_{\leq N(t_0)}\|_{L_{t,x}^\delta([t_0, t_0 + \frac{\delta}{N(t_0)^3}] \times \mathbf{R})}^6 \\ &\lesssim \frac{1}{N(t_0)} + N(t_0)^2 \frac{1}{N(t_0)^3} \sim \int_{t_0}^{t_0 + \frac{\delta}{N(t_0)^3}} N(t)^2 dt. \end{aligned} \quad (7.45)$$

Therefore, by conservation of energy

$$\frac{R_1}{R} \int_I \int \int u(t, y)^6 u_x^2 dx dy dt \lesssim \frac{R_1}{R} \int_I N(t)^2 dt. \quad (7.46)$$

Next, let  $J_l = [t_0, t_0 + \frac{\delta}{N(t_0)^3}]$ .

$$\frac{R_1}{R} \frac{\tilde{N}(t)^3}{R_1^2} \int_{J_l} \left( \int_{|x-y| \lesssim \frac{R}{N(t)}} u_x^2 u^2 \right) \quad (7.47)$$

$$\lesssim \frac{\tilde{N}(t)^3}{R_1 R} \|u_x(u^2)\|_{L_{t,x}^2(J_l \times \mathbf{R})} \|u\|_{L_t^\infty L_x^2(J_l \times \mathbf{R})} \frac{R}{\tilde{N}(t_0)} \frac{1}{N(t_0)^{3/2}} \lesssim \frac{1}{R_1} \frac{\tilde{N}(t_0)^2}{N(t_0)^{3/2}}. \quad (7.48)$$

The last inequality follows from conservation of energy, Holder's inequality, and

$$\|u\|_{L_x^4 L_t^\infty(J_l \times \mathbf{R})} \lesssim \|\partial_x u\|_{S^0(J_l \times \mathbf{R})}^{1/4} \|u\|_{S^0(I \times \mathbf{R})}^{3/4}, \quad (7.49)$$

$$\|u\|_{L_x^\infty L_t^2(J_l \times \mathbf{R})} \lesssim \|u\|_{S^0(I \times \mathbf{R})}. \quad (7.50)$$

Next, by conservation of mass

$$\frac{\tilde{N}(t_0)^4}{R R_1^3} \int_{J_l} \int_{|x-y| \sim \frac{R}{N(J)}} u(t, x)^2 u(t, y)^2 dx dy \lesssim \frac{\tilde{N}(t_0)}{R R_1^3}. \quad (7.51)$$

Finally, by conservation of mass and (7.45)

$$\int_I \frac{\tilde{N}(t)^3}{R_1 R} \int u(t, x)^2 u(t, y)^6 dx dy dt \lesssim \frac{1}{R_1 R} \int \tilde{N}(t)^2 N(t)^2 dt. \quad (7.52)$$

This takes care of the error terms in (7.34), (7.36), (7.39), and (7.40). By the fundamental theorem of calculus and the above computations, taking say  $R_1 = R^{1/2}$ ,

$$\int_I N(t)^4 \tilde{N}(t) dt \lesssim \eta(R) \int_I N(t)^2 \tilde{N}(t) dt + R \int \frac{|\tilde{N}'(t)|}{\tilde{N}(t)} \int_{|x-y| \lesssim \frac{R}{\tilde{N}(J)}} u_x^2 u^2 dx dy dt, \quad (7.53)$$

where  $\eta(R) \rightarrow 0$  as  $R \rightarrow \infty$ . Now choose  $\tilde{N}(t) = N(t)$  for  $N(t) \leq \alpha$  and  $\tilde{N}(t) = \alpha$  for  $N(t) \geq \alpha$ ,  $\alpha > 0$  some small fixed constant.

$$R \int_{J_I} \tilde{N}(t_0)^3 \int_{|x-y| \lesssim \frac{R}{\tilde{N}(t_0)}} u_x^2 u^2 dx dy dt \lesssim \frac{R^{3/2}}{\tilde{N}(t_0)^{3/2}} \tilde{N}(t_0)^2 \sim R^{3/2} \int_{J_I} \tilde{N}(t)^2 N(t)^{3/2} dt. \quad (7.54)$$

Since  $\tilde{N}(t) \leq \alpha$ ,  $\tilde{N}(t) \leq N(t)$ ,

$$R^{3/2} \int_I \tilde{N}(t)^2 N(t)^{3/2} \lesssim R^{3/2} \alpha^{3/2} \mathcal{J}. \quad (7.55)$$

$$\eta(R) \int_I \tilde{N}(t) N(t)^2 dt \lesssim \eta(R) \alpha \mathcal{J}. \quad (7.56)$$

Next,

$$\int_{t: N(t) \leq \alpha} N(t)^4 dt \leq \alpha^2 \mathcal{J}. \quad (7.57)$$

Since  $\int_I N(t)^4 dt \gtrsim \mathcal{J}$ , by the fundamental theorem of calculus and the error estimates,

$$\alpha \mathcal{J} \lesssim \alpha \int_I N(t)^4 dt \lesssim R + \eta(R) \alpha \mathcal{J} + R^{3/2} \alpha^{3/2} \mathcal{J}. \quad (7.58)$$

Choose  $\alpha(R)$  sufficiently small so that  $\alpha^{3/2} R^{3/2} \ll \eta(R)$ . Then for  $\mathcal{J}$  sufficiently large, we have a contradiction.  $\square$

## References

- [1] T. Cazenave and F. B. Weissler, two authors "The Cauchy problem for the critical nonlinear Schrödinger equation in  $H^{s^*}$ ", *Nonlinear Anal.*, **14** (1990), 807–836.
- [2] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao. "Global existence and scattering for rough solutions of a nonlinear Schrödinger equation on  $\mathbf{R}^3$ " *Communications on pure and applied mathematics*, **21** (2004) : 987 - 1014

- [3] B. Dodson, *Global well - posedness and scattering for the defocusing  $L^2$  - critical nonlinear Schrödinger equation when  $d = 1$* , preprint, *arXiv:1010.0040v2*,
- [4] B. Dodson, *Global well-posedness and scattering for the mass critical nonlinear Schrödinger equation with mass below the mass of the ground state*, preprint, *arXiv:1104.1114v2*,
- [5] M. Hadac and S. Herr and H. Koch, “Well-posedness and scattering for the KP-II equation in a critical space”, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **26** (2009): 3, 917–941.
- [6] T. Kato, “On the Cauchy problem for the (generalized) Korteweg-de Vries equation” *Studies in applied mathematics (Adv. Math. Suppl. Stud.)* **8** (1983): 93 – 128.
- [7] C. E. Kenig, G. Ponce, and L. Vega “Oscillatory integrals and regularity of dispersive equations”, *Indiana University Mathematics Journal* **40** 1 (1991): 33 – 69.
- [8] C. E. Kenig, G. Ponce, and L. Vega “Well - posedness and scattering results for the generalized Korteweg-de Vries equation via the contraction principle”, *Communications on Pure and Applied Mathematics* **46** 4 (1993): 527 – 620.
- [9] C. E. Kenig, G. Ponce, and L. Vega “On the concentration of blow up solutions for the generalized KdV equation critical in  $L^2$ ”, *Nonlinear Wave equations* (Providence, RI, 1998), Contemporary Mathematics. *American Mathematical Society, Providence, RI* **263** (2000): 131 – 156.
- [10] R. Killip, S. Kwon, S. Shao, and M. Visan, “On the mass - critical generalized KdV equation”, *Discrete and Continuous Dynamical Systems. Series A* **32** 1 (2012): 191 – 221.
- [11] R. Killip, T. Tao, and M. Visan “The cubic nonlinear Schrödinger equation in two dimensions with radial data” *Journal of the European Mathematical Society* , to appear.
- [12] R. Killip, M. Visan, and X. Zhang, “The mass-critical nonlinear Schrödinger equation with radial data in dimensions three and higher” *Annals in PDE* , textbf1, no. 2 (2008) 229 - 266
- [13] S. Kwon and S. Shao, *Nonexistence of Soliton - Like Solutions for Defocusing Generalized KDV Equations*, preprint, *arXiv:1205.0849*,
- [14] E. M. Stein, “Harmonic Analysis: Real-variable Methods, Orthogonality, and Oscillatory Integrals,” Princeton University Press, Princeton, NJ, 1993.
- [15] T. Tao, “Nonlinear Dispersive Equations,” Published for the Conference Board of the Mathematical Sciences, Washington, DC, 2006.
- [16] T. Tao, “Two Remarks on the Generalised Korteweg de - Vries Equation”, *Discrete and Continuous Dynamical Systems. Series A* **18** 1 (2007): 1 – 14.

- [17] T. Tao, M. Visan, and X. Zhang. "Minimal-mass blowup solutions of the mass-critical NLS." *Forum Mathematicum*, **20** no. 5 (2008) : 881 - 919.
- [18] T. Tao, M. Visan, and X. Zhang. "Global well-posedness and scattering for the defocusing mass - critical nonlinear Schrödinger equation for radial data in high dimensions." *Duke Mathematical Journal*, **140** no. 1 (2007) : 165 - 202.